Last Time: Change of Basis!

Represent a linear map $L : V \to W$ via many matrices...

Special Cases: If $V = W$ and $L = id$.

$\text{Rep}_{B,D}(id)$ is the matrix representing the change of basis B to D.

$$
\begin{tikzcd}
V_B \arrow[r, "\text{Rep}_{B,D}(L)"] \arrow[r, shift right=2] \arrow[d, shift right, "\text{Rep}_{B,B'}(id)"'] & W_D \arrow[d, shift left, "\text{Rep}_{D,D'}(id)"] \arrow[d, shift right=2] \\
V_{B'} \arrow[r, "\text{Rep}_{B',D'}(L)"'] \arrow[r, shift left=2] \arrow[u, bend right, shift right=2, "\text{Rep}_{B',B}(id)"'] & W_{D'}
\end{tikzcd}
$$

$$\text{Rep}_{B',D'}(L) = \underline{R_{D,D'}(id)} \cdot \underline{\text{Rep}_{B,D}(L)} \cdot \underline{\text{Rep}_{B',B}(id)}$$

WHY?: Some bases make for really simple representations of your linear map...

Remark: Some "nice" linear operators can be represented by diagonal matrices...

Ex: Consider the spaces $V = \mathcal{P}_2(\mathbb{R})$ and $W = \mathcal{M}_{2\times 2}(\mathbb{R})$.

$$B = \{1,\ 1+x,\ 1+x^2\}, \quad B' = \{1,\ x,\ x^2\} \subseteq V$$

$$D = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$D' = \left\{ \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

(A) Build $\mathrm{Rep}_{\underline{B,B'}}(id_V)$ and $\mathrm{Rep}_{D,D'}(id_W)$.

$$[B' \mid B] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \underline{0} & \underline{0} & \underline{1} \end{array}\right] \rightsquigarrow \underline{\mathrm{Rep}_{B,B'}(id_V) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$[D' \mid D] \rightsquigarrow \left[\begin{array}{cccc|cccc} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\therefore \mathrm{Rep}_{D,D'} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

(B) Suppose $L : V \to W$ is represented by

$$\mathrm{Rep}_{B,D}(L) = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}. \qquad \text{What is } \mathrm{Rep}_{B',D'}(L)\,?$$

$$\begin{tikzcd}
V_B \arrow[r, "\mathrm{Rep}_{B,D}(L)"] \arrow[d, "\mathrm{Rep}_{B,B'}(id)"'] & W_D \arrow[d, "\mathrm{Rep}_{D,D'}(id)"] \\
V_{B'} \arrow[r, "\mathrm{Rep}_{B',D'}(L)"'] \arrow[u, bend right] & W_{D'}
\end{tikzcd}$$

$$\therefore \operatorname{Rep}_{B',D'}(L) = \operatorname{Rep}_{D,D'}(id) \cdot \operatorname{Rep}_{B,D}(L) \cdot \operatorname{Rep}_{B',B}(id)$$

Now we compute $\operatorname{Rep}_{B',B}(id) = \operatorname{Rep}_{B,B'}(id)^{-1}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & \boxed{1} & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$\therefore \operatorname{Rep}_{B',B}(id) = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so finally:

$$\operatorname{Rep}_{B',D'}(L) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}$$

Eigenvectors and Eigenvalues

Goal: Understand when a matrix can be diagonalized...

↳ On hold... we'll build up to this ⌣̈

Defn: Let $L: V \to V$ be a linear operator.

① An *eigenvector* of L is an element $v \in V$ such that $L(v) = \lambda v$ for some scalar λ.

② The eigenvalue of eigenvector $v \in V$ for L is the scalar λ with $L(v) = \lambda v$.

<u>More succinctly</u>: An eigenvector of L w/ eigenvalue λ is a vector $v \in V$ with $L(v) = \lambda v$. ☆

<u>NB</u>: "eigen" means (roughly) "same" in German.

<u>Ex</u>: Consider the transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x \\ 5y \\ 0 \end{pmatrix}$. Note that

$L(e_1) = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = 3e_1$ So e_1 is an eigenvector of L with eigenvalue 3.

$L(e_2) = 5e_2$ So e_2 is an eigenvector of L w/ eigenvalue 5.

$L(e_3) = \vec{0}$ So e_3 is a eigenvector w/ eigenvalue 0...

$L(\vec{0}) = \vec{0} = \lambda \vec{0}$ for all $\lambda \in \mathbb{R}$...

<u>But</u> for technical reasons, we do NOT call $\vec{0}$ an eigenvector...

<u>Remark</u>: v is an eigenvector w/ eigenvalue 0 if and only if <u>$v \in \ker(L)$</u>.

↳ <u>Exercise</u>: prove it!

<u>Prop</u>: If v, w are eigenvectors of L w/ eigenvalue λ, then ① av also has eigenvalue λ.
② $v + w$ also has eigenvalue λ.

Q: How do I compute eigenvalues and eigenvectors?

Note: $L(v) = \lambda v$ if L is represented by

$\mathrm{Rep}_{B,B}(L) = M$, then we're asking for:

$$M u = \lambda u = \lambda I_n u$$

so $$M u - \lambda I_n u = 0$$

i.e. $$\underbrace{(M - \lambda I_n)}\, u = 0$$

So this transformation has u in its kernel...

Thus $\underline{\det(M - \boxed{\lambda} I_n)} = 0$...

Def^n: The characteristic polynomial of matrix M (or more generally the operator associated to M) is the polynomial $p_M(\lambda) := \det(M - \lambda I)$.

Point: Every eigenvalue of M is a root of the characteristic polynomial $p_M(\lambda)$.

Ex: Compute $p_M(\lambda)$ for $M = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.

Sol: $p_M(\lambda) = \det(M - \lambda I)$

$$= \det \begin{bmatrix} 1-\lambda & 0 & 1 \\ 1 & 1-\lambda & -1 \\ 0 & 1 & -\lambda \end{bmatrix} = (1-\lambda)\det\begin{bmatrix} 1-\lambda & -1 \\ 1 & -\lambda \end{bmatrix} - \det\begin{bmatrix} 0 & 1 \\ 1 & -\lambda \end{bmatrix} + 0$$

$$= (1-\lambda)\big(-\lambda(1-\lambda) + 1\big) - (-1)$$

$$= (1-\lambda)(1 - \lambda + \lambda^2) + 1$$

$$= (1-\lambda+\lambda^2) - \lambda(1-\lambda+\lambda^2) + 1$$

$$= 1-\lambda+\lambda^2-\lambda+\lambda^2-\lambda^3+1$$

$$= -\lambda^3+2\lambda^2-2\lambda+2$$

$\boxed{}$

Ex: Consider $L\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}5x\\-y\\0\end{pmatrix}$. This transformation has matrix $M = \begin{bmatrix}5&0&0\\0&-1&0\\0&0&0\end{bmatrix}$ wrt $\mathcal{E}_3$. We

Compute $p_M(\lambda) = \det(M-\lambda I)$

$$= \det\begin{bmatrix}5-\lambda&0&0\\0&-1-\lambda&0\\0&0&-\lambda\end{bmatrix} = (5-\lambda)(-1-\lambda)(-\lambda)$$

$$= \lambda(\lambda+1)(5-\lambda).$$

which has roots $\lambda=0$, $\lambda=-1$, and $\lambda=5$. $\boxed{}$